Improved Approximation for Tree Augmentation via Chvátal Gomory Cuts

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Joint work with S. Fiorini, M. Groß, and L. Sanità

COMBINATORICS & OPTIMIZATION

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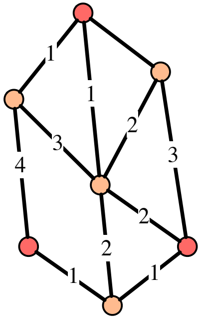
FACULTY OF MATHEMATICS

Motivation – Network Design

Input: graph $G = (V, E)$ with non-negative edge-weights w_e for all edges $e \in E$



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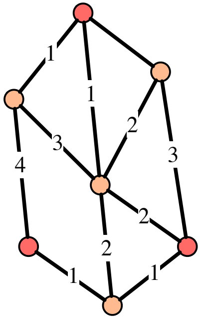

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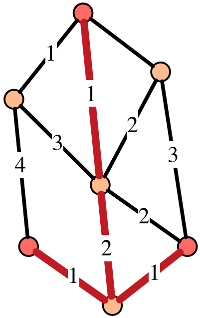

Steiner tree: find min-weight tree that spans a given set of special vertices

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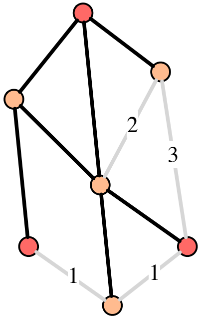

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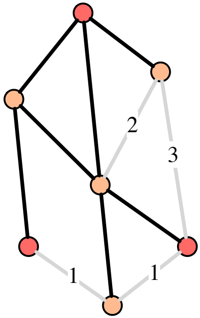

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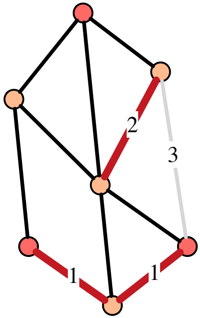

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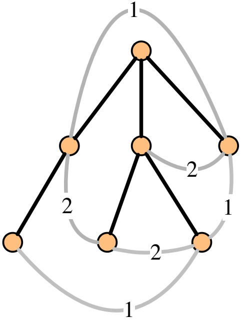
Weighted Tree Augmentation (WTAP):

- Input: Tree $T = (V, E)$, links $L \subseteq V \times V$, non-negative weight w_l for each $l \in L$



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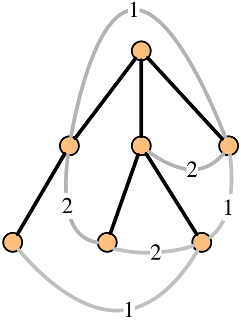

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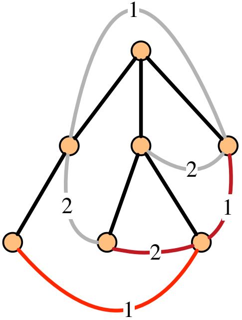

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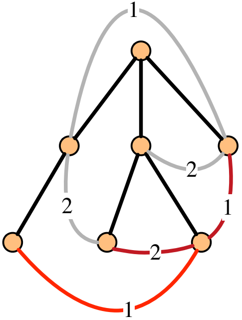
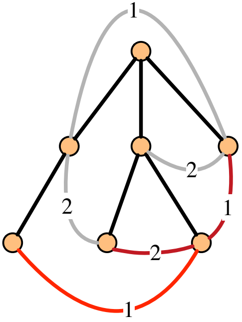


Unit-weight tree augmentation (TAP): special case where $w_l = 1$ for all links l

TAP is Hard

[Fredrickson and Jájá '81] WTAP and TAP are NP-hard even if the tree has constant diameter.



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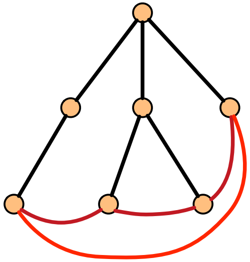
[Cheriyan, Jordán, Ravi '99] TAP is NP-hard even if the links form a cycle on the leaves of the tree.



Approximation Algorithms

For a given instance (T, w) of WTAP, we use opt for the weight of an optimum feasible solution.

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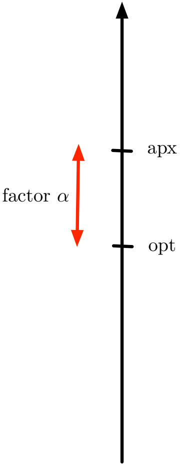
An efficient algorithm $\mathcal{A}$ is an α-approximation for WTAP if it produces a link set of weight no more than $\alpha\,$opt, for all instances.



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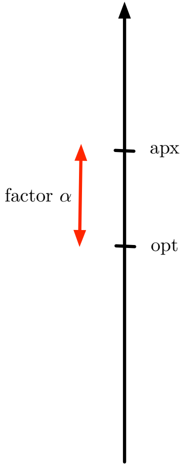

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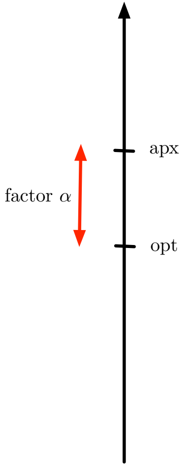
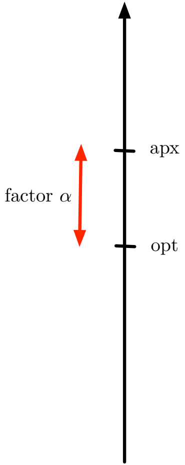
$\longrightarrow$ there is a constant α such that no $(\alpha - \epsilon)$-approximation exists for any $\epsilon > 0$ unless P=NP.



Approximation Algorithms via LP

Popular approach:

- Formulate the given problem as mathematical program



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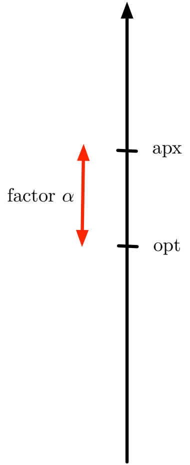
- Formulate the given problem as mathematical program
- Relax it, so it can be solved efficiently



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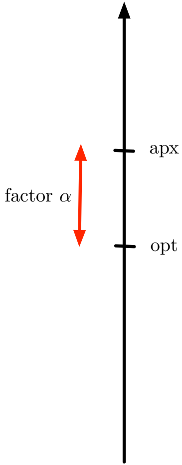

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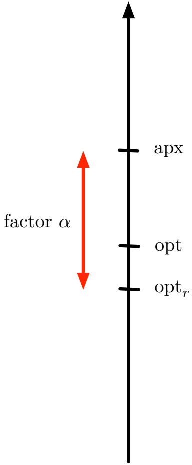
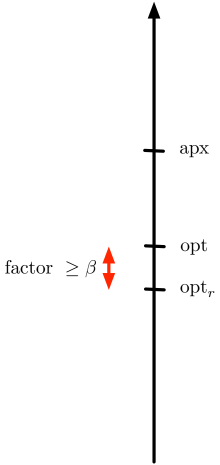
- Let opt_r be the optimum value of the relaxation, and show that the solution has cost at most $\alpha\,\mathrm{opt}_r$



Approximation Algorithms via LP

The maximum ratio of opt and opt_r over all instances of the problem is called the integrality gap of the relaxation.



Approximation Algorithms via LP

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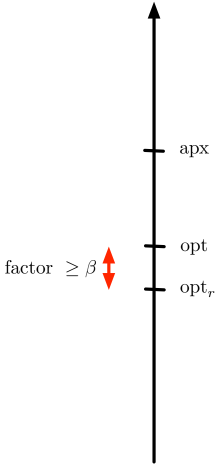
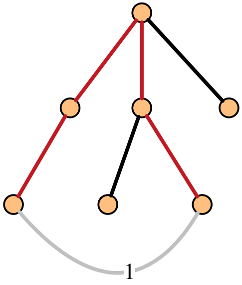
Note: if the integrality gap of a relaxation is at least β then $\alpha \geq \beta$ for any α-approximation that uses this relaxation



An IP for Tree Augmentation

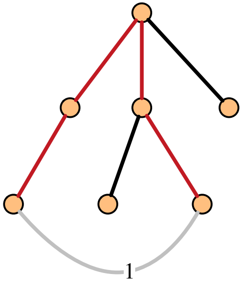
- For link $l \in L$, let $T(l)$ be the unique path in T connecting the endpoints of l



An IP for Tree Augmentation

- For link $l \in L$, let $T(l)$ be the unique path in T connecting the endpoints of l
- For edge $e \in E$, let cov(e) be the set of links $l \in L$ that cover e; i.e.,

$$\text{cov}(e) = \{l \in L : e \in T(l)\}$$



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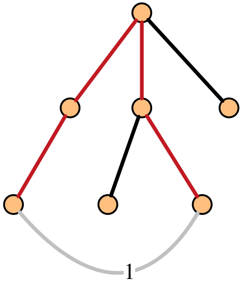

Folklore

$S \subseteq L$ is a feasible solution for TAP iff $S \cap \text{cov}(e) \neq \emptyset$, for all $e \in E$

An IP for Tree Augmentation

- Introduce indicator variable x_l for each $l \in L$
- LP relaxation of IP:

$$
\begin{aligned}
\min \quad & \sum_{l} w_l x_l & \text{(P)} \\
\text{s.t.} \quad & \sum_{l \in \mathsf{cov}(e)} x_l \geq 1 \quad (e \in E) \\
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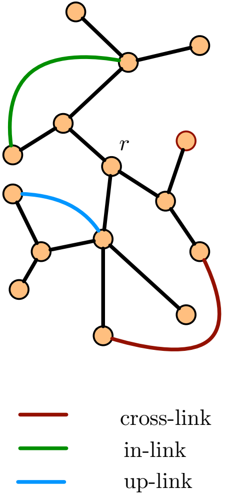
In what follows we let A be the coefficient matrix for the above LP, and thus (P) can be rewritten as

$$\min\{w^T x \,:\, Ax \geq \mathbb{1}, x \geq \mathbb{0}\}$$

Easy WTAP Instances

- Pick an arbitrary fixed root $r \in V$



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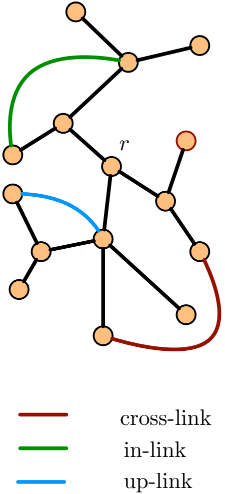

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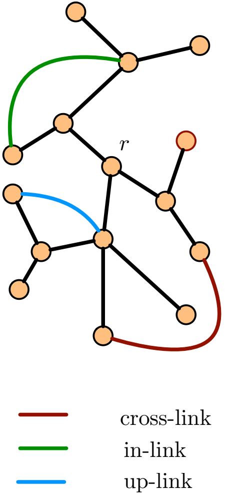

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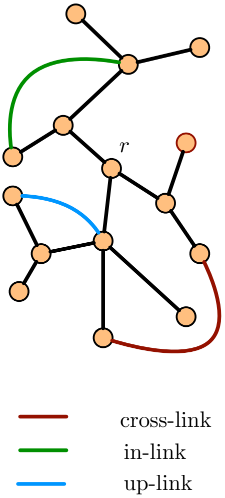

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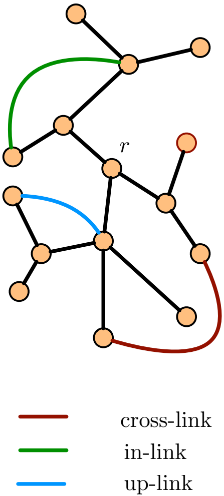
Coefficient matrix A of LP (P) is network matrix for up-link-only WTAP instances.



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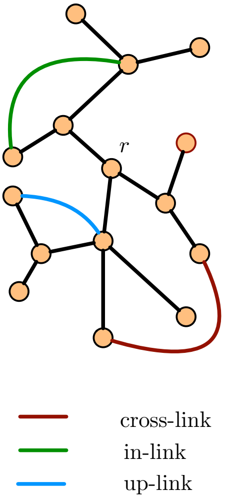

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$\rightarrow$ (P) is integral in "up-link only" instances.

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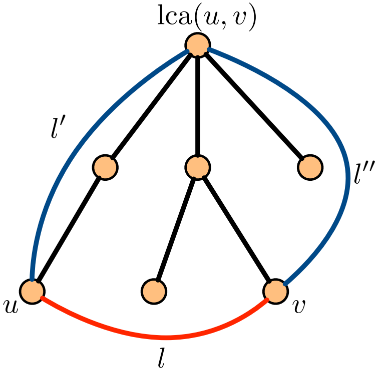
- Given an instance (T, w) of WTAP, pick an arbitrary root $r \in V(T)$



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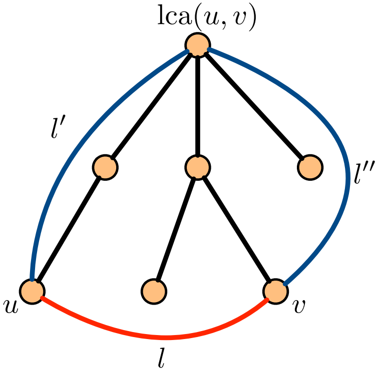

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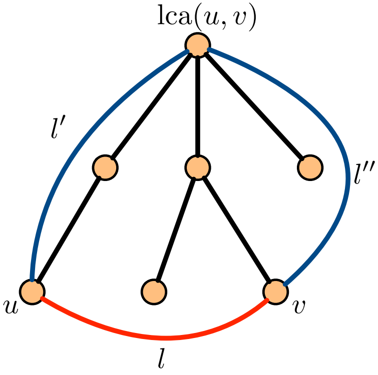
- Obtain a new instance (T, w') by replacing each cross link $l = (u, v)$ by two up-links $(u, \text{lca}(u, v))$ and $(v, \text{lca}(u, v))$ of the same cost



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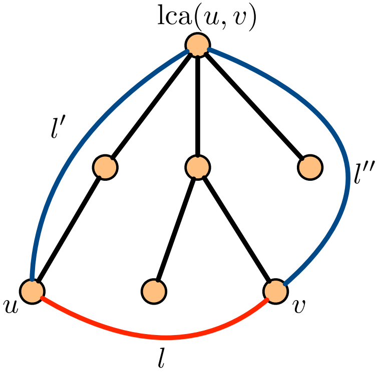

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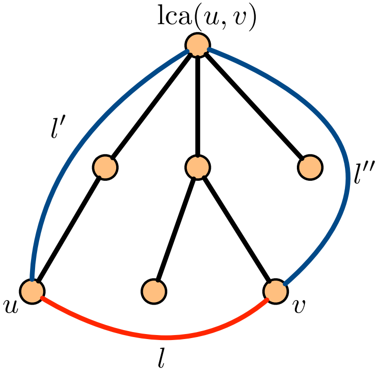
- Note: $\mathsf{opt}_{T,w'} \leq 2\,\mathsf{opt}_{T,w}$



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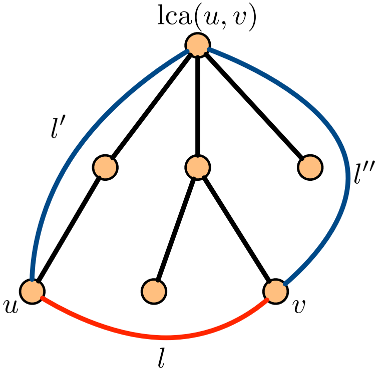
- Gives a 2-approximation for WTAP, and this is best known!



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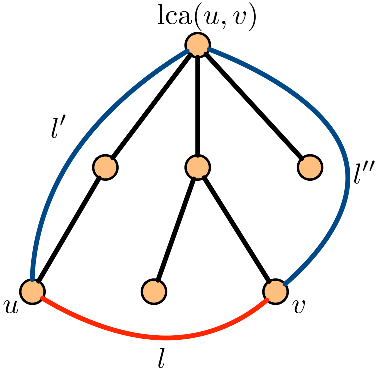

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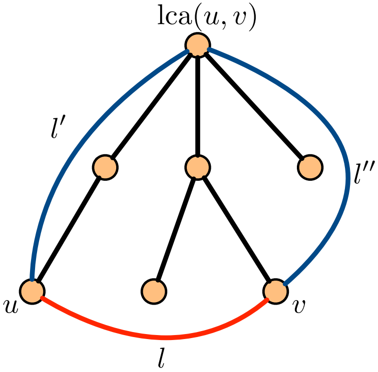
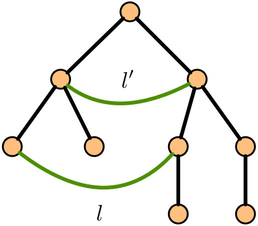


*Shadow Completeness

It is sometimes convenient to assume that our WTAP instances are shadow complete:

$$l \in L \quad \longrightarrow \quad l' \in L,$$

whenever l' spans a subset of l's edges of T



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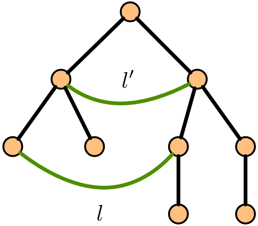
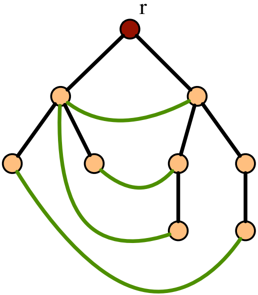
Clearly: assumption is w.l.o.g.



Star Shaped Instances

- A WTAP instance (T, w) is star shaped if all links $l \in L$ cover r for some (arbitrary) root node $r \in V$



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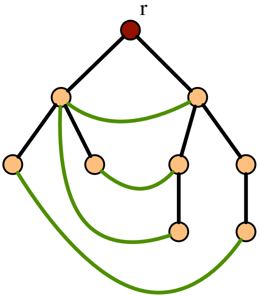

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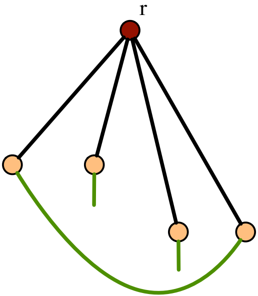
- WTAP instance is edge-cover in disguise: pick a minimum-weight collection of links that covers all leaf vertices



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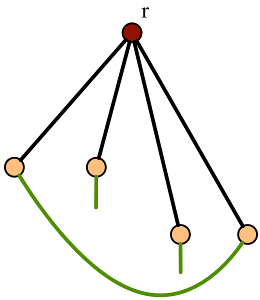
- Edge-cover is polynomial-time solvable $\rightarrow$ Star shaped WTAP instances are polynomial-time solvable



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- (P) for such instances is equivalent to the fractional edge-cover LP

$$
\begin{aligned}
\min \quad & c^T x \\
\text{s.t.} \quad & \sum_{l \in \mathsf{cov}(e)} x_l \geq 1 \\
& \text{(leaf edges } e) \\
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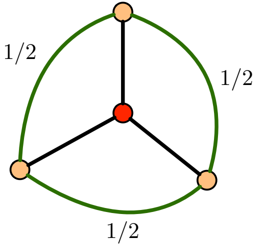

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- There is an exact, tractable LP for edge-cover

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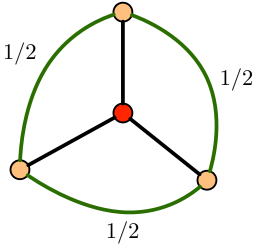
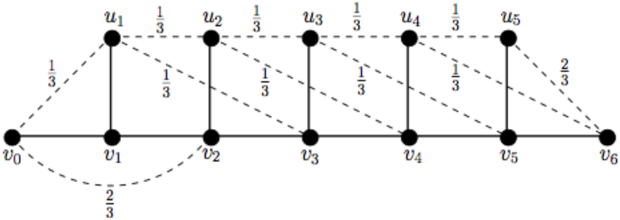


Recall: Tree Augmentation Formulation

- Introduce indicator variable x_l for each $l \in L$
- LP relaxation of IP:

$$\begin{aligned} \min \quad & \sum_{l} w_l x_l \\ \text{s.t.} \quad & \sum_{l \in \mathsf{COV}(e)} x_l \geq 1 \quad (e \in E) \\ & x \geq \mathbb{0} \end{aligned} \tag{P}$$

Integrality Gap Example for (P)



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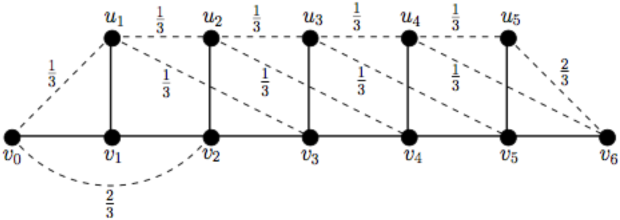

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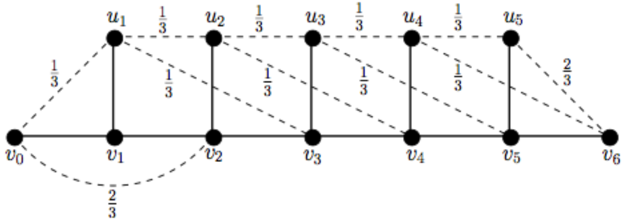
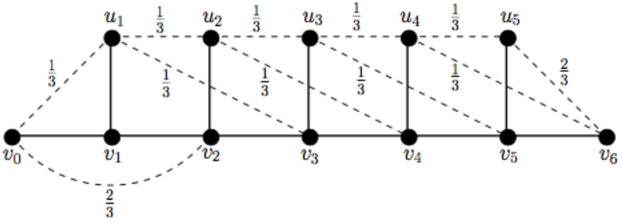


- the figure shows tree T_k for $k = 5$ and unit weight links (dashed)
- The edge-labels of the links give a solution x for (P) of value $2k/3 + 1 = 13/3$
- The best integral solution has value $k + 1 = 6$.
  $\rightarrow$ the integrality gap of (P) is $\approx 3(k + 1)/2k$ and thus tends to $3/2$ for large k

Integrality Gap Example for (P)



Theorem [Cheriyan, Karloff, Khandekar, K. '08]

LP (P) has integrality gap at least $3/2$.

TAP/WTAP Status Quo

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  Comment: Adjiashvili's results are with respect to a stronger TAP LP

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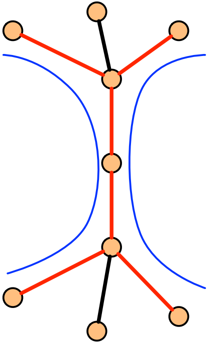
- (Fairly) simple algorithms and analysis

Adjiashvili's Framework

Strengthening the LP

- Tree $T = (V, E)$, and constant γ (later chosen to be $O(M/\epsilon^2)$ for small $\epsilon > 0$)



2-bundle

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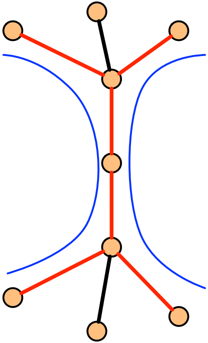

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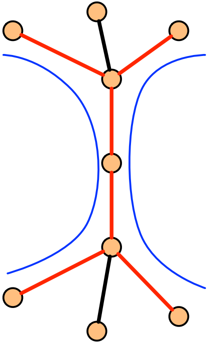

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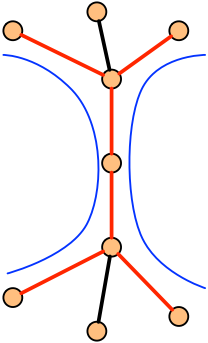

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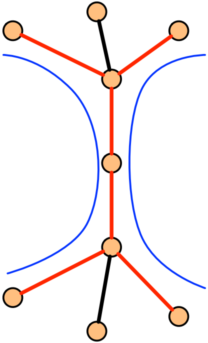
- opt_B: optimum weight of a WTAP solution for γ-bundle B



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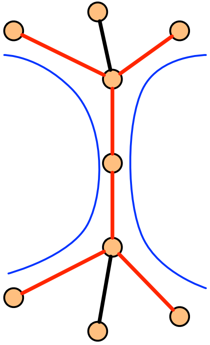
Let B be a forest with at most k leaves.
$\rightarrow$ can compute opt$_B$ in time $|V|^{k^{O(1)}}$.



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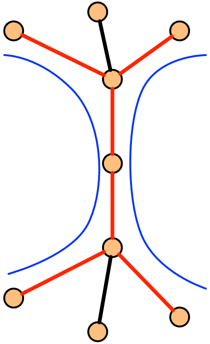

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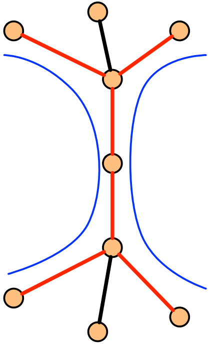
Strengthen (P) by adding constraints:

$$\sum_{l \in \mathsf{cov}(B)} w_l x_l \geq \mathsf{opt}_B \quad (B \in B_\gamma)$$

B_γ : set of γ-bundles



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$$\min \quad \sum_{l} w_l x_l \tag{P_1}$$

$$\text{s.t.} \quad \sum_{l \in \mathsf{cov}(e)} x_l \geq 1 \quad (e \in E)$$

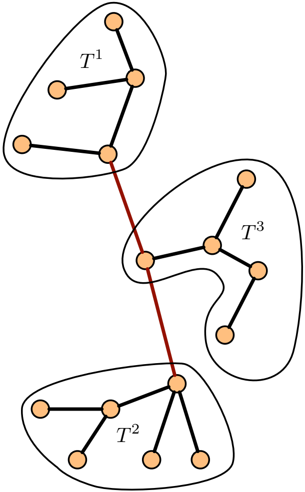
$$\sum_{l \in \mathsf{cov}(B)} w_l x_l \geq \mathsf{opt}_B \quad (B \in B_\gamma)$$

$$x \geq \mathbb{0}$$

Consequence from earlier discussion: can solve (P_1) in polynomial time for fixed γ.

Birdseye View of Adjiashvili

- Goal: round solution x to P_1.



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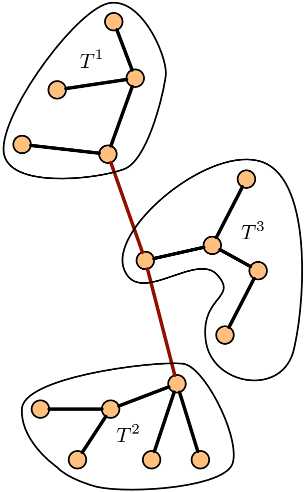

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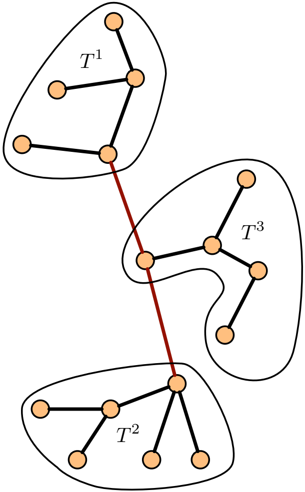

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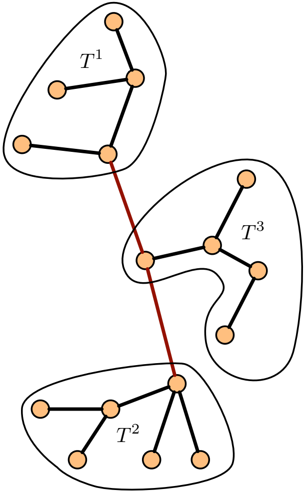

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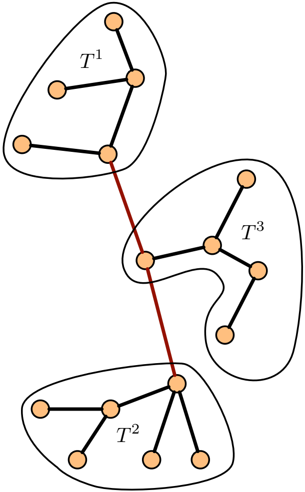
  (iii) structure of T^i is easier



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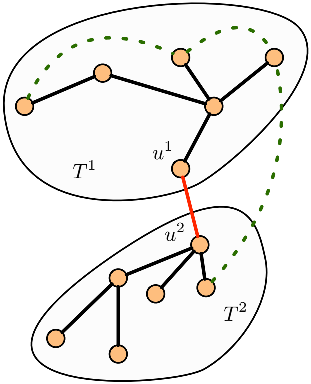

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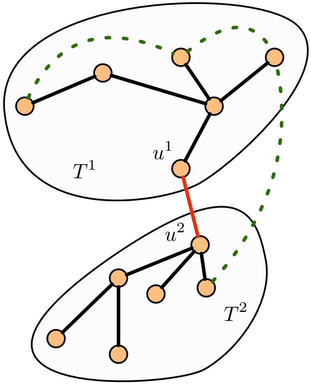

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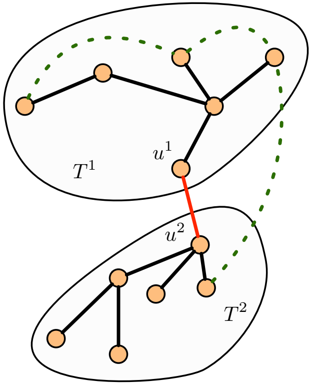

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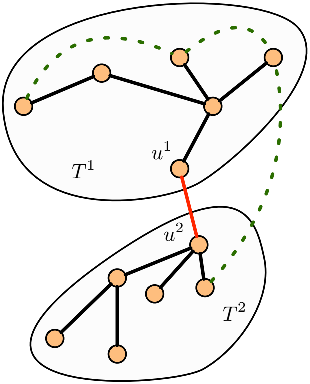

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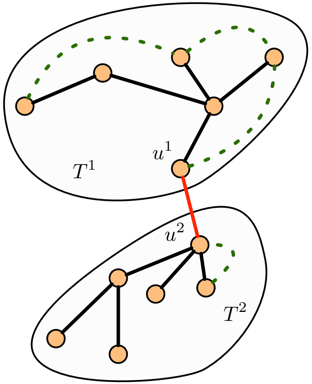

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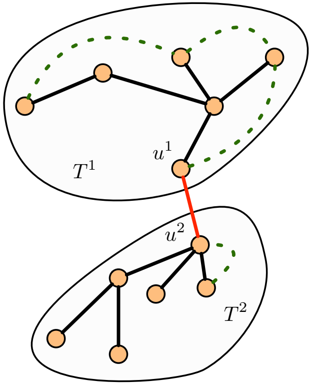

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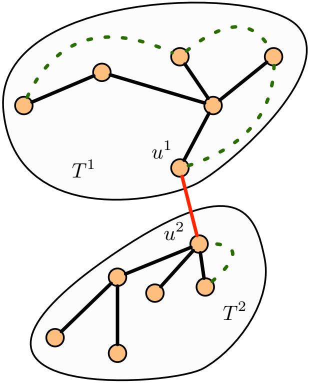

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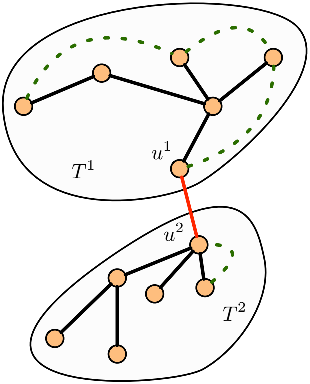
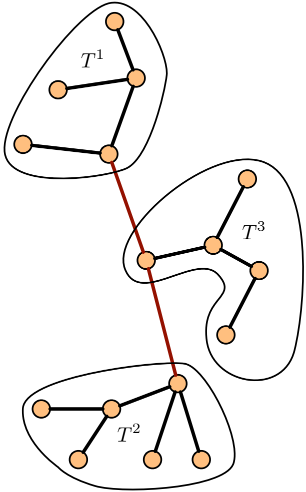
- Can now charge weight increase due to x-splitting to x-weight of links in the decomposition parts



Birdseye View of Adjiashvili

Repeated decomposition creates

- subtrees $T^1, T^2, \ldots, T^q$, and
- independent solutions $x^1, x^2, \ldots, x^q$ for (P_1)



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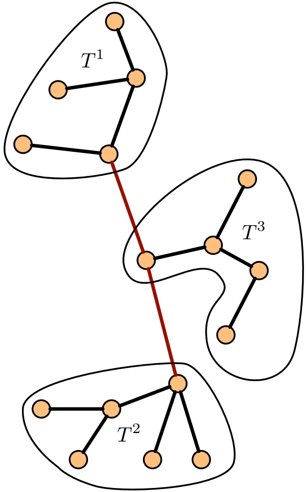
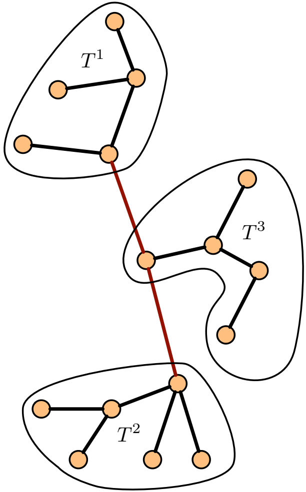
- Round feasible solution x^i for each $1 \leq i \leq k$ independently



Birdseye View of Adjiashvili

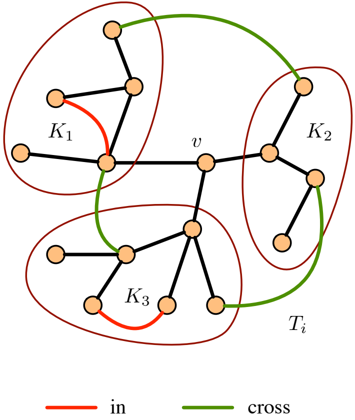
- Round feasible solution x^i for each $1 \leq i \leq k$ independently
- Return the union of solutions for sub-instances



Birdseye View of Adjiashvili

Each (T^i, x^i) will turn out to be β-simple:

- There is a center vertex $v \in V(T^i)$ whose removal breaks T^i into subtrees $K_1, \ldots, K_q$



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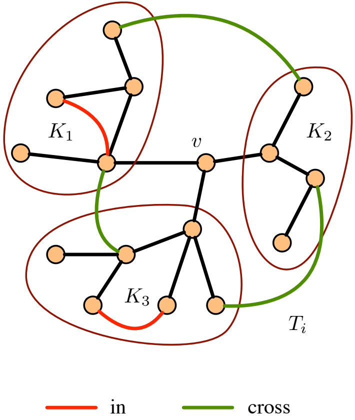

Birdseye View of Adjiashvili

Each (T^i, x^i) will turn out to be β-simple:

- There is a center vertex $v \in V(T^i)$ whose removal breaks T^i into subtrees $K_1, \ldots, K_q$
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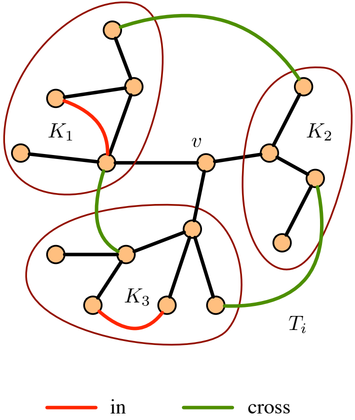

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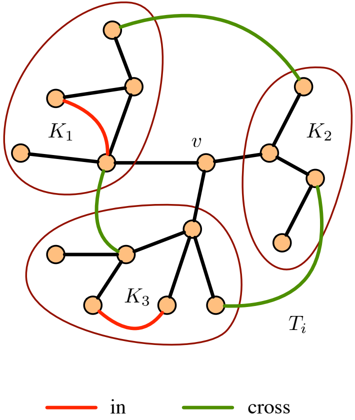

Solving Sub-instance (T^i, x^i)

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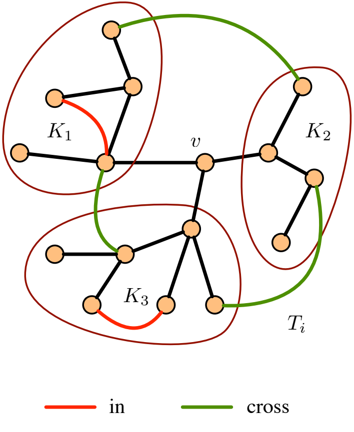

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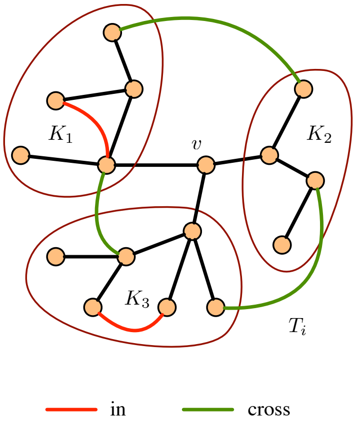

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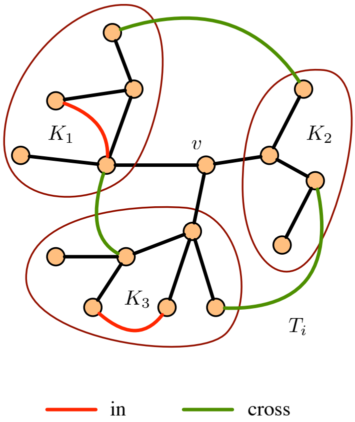

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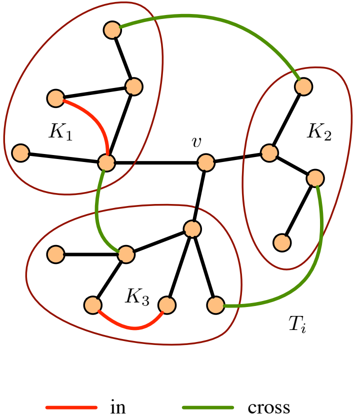

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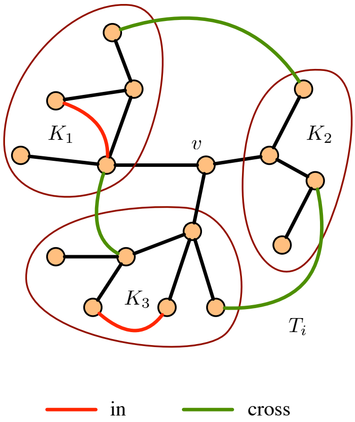
- Will use the better of two rounding methods

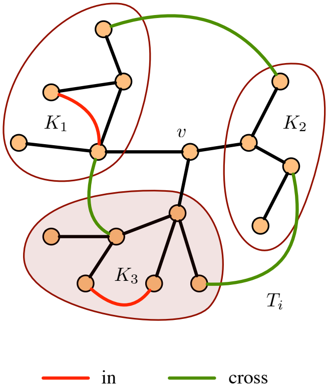


Rounding Method 1 – Inlink-Heavy Case

Theorem [Adjiashvili '17]

There is an algorithm that computes a set S of links covering T^i such that

$$c(S) \leq \sum_{l \in \mathcal{I}} w_l x_l^i + 2 \sum_{l \in \mathcal{C}} w_l x_l^i$$



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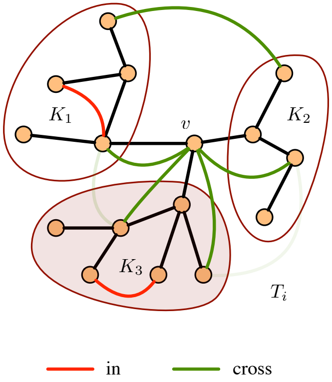
- Create new solution y from x^i by splitting each cross-link at center v;



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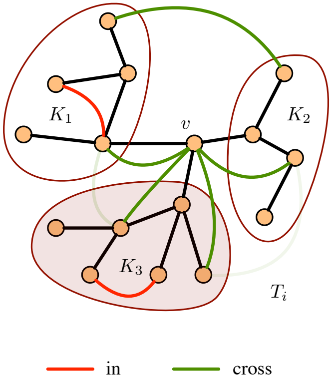

Sketch:

- Create new solution y from x^i by splitting each cross-link at center v; easy to see:

$$w^T y \leq \sum_{l \in \mathcal{I}} w_l x_l^i + 2 \sum_{l \in \mathcal{C}} w_l x_l^i$$

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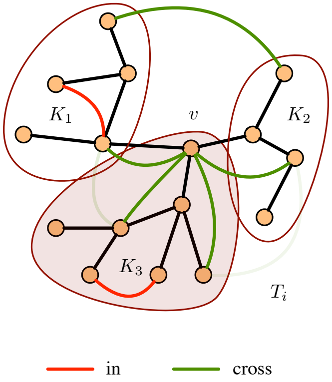
- Add center v to instance $\rightarrow \overline{K}^i$



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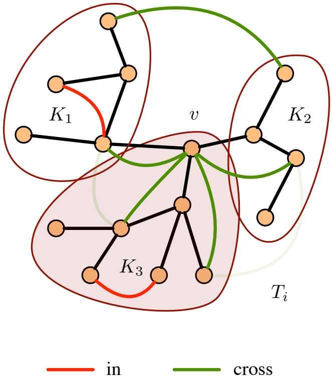

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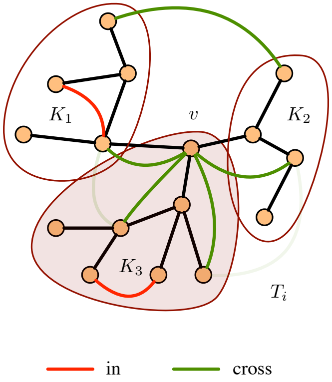

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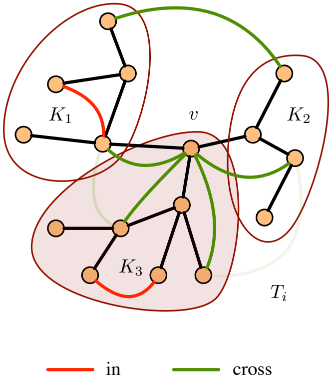
- Implies: fractional cost of y is as large as optimum solution for $\overline{K}^i$ (which can be computed efficiently).



Rounding Method 2 – Crosslink-Heavy Case

Theorem [Adjiashvili '17]

There is an algorithm that computes a set S of links covering T^i such that

$$c(S) \leq 2\lambda \sum_{l \in \mathcal{I}} w_l x_l^i + \frac{4\lambda}{3(\lambda - 1)} \sum_{l \in \mathcal{C}} w_l x_l^i$$

for any constant $\lambda > 1$.

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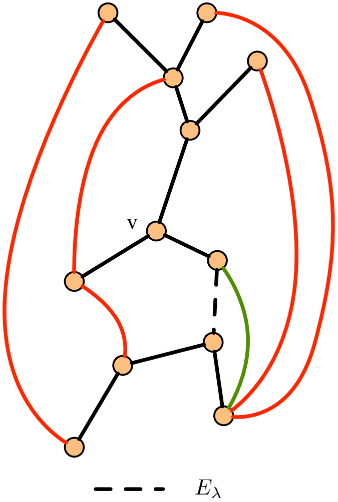
- Can cover edges in E_λ at cost no larger than $2\lambda \sum_{l \in \mathcal{I}} w_l x_l^i$

Rounding Method 2 – Crosslink-Heavy Case

- Contract all edges in $E_\lambda \rightarrow \bar{T}^i$, and delete all inlinks



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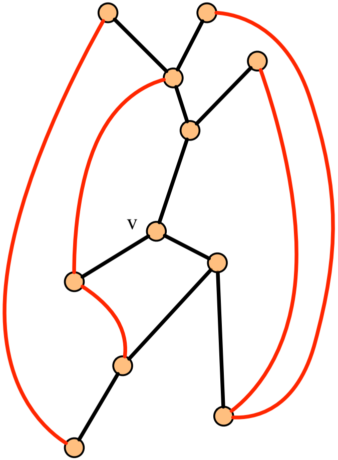

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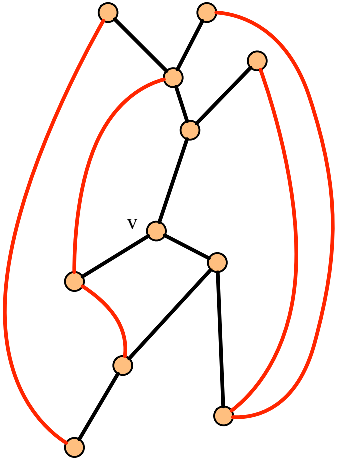

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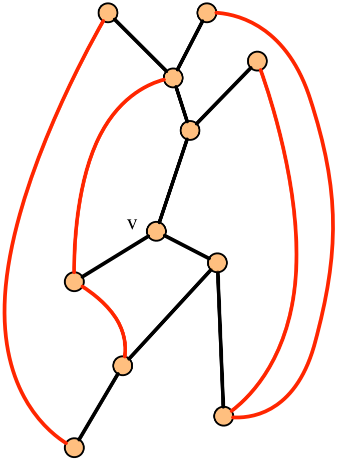

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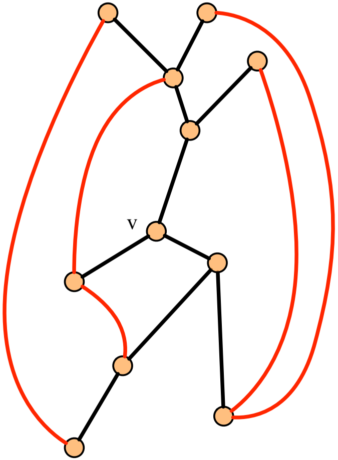
  - (ii) $z = \lambda/(\lambda-1) \cdot x^i$ is now feasible for original TAP LP (P)
- Hence: there is a solution $S \subseteq L$ for this instance of cost no more than

$$\frac{4\lambda}{3(\lambda-1)} \sum_{l \in \mathcal{C}} w_l x_l^i$$



Adjiashvili's Method: Wrapping Up

- Have seen: for the instance induced by T^i, can compute feasible solution S of links of cost

$$\min\{\sum_{l\in\mathcal{I}} w_l x_l^i + 2\sum_{l\in\mathcal{C}} w_l x_l^i, 2\lambda\sum_{l\in\mathcal{I}} w_l x_l^i + \frac{4\lambda}{3(\lambda-1)}\sum_{l\in\mathcal{C}} w_l x_l^i\}$$

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Theorem [Adjiashvili '17]

Choosing $\lambda = 3 + \sqrt{5}$ and simple numerical optimization yields that algorithm is $(1.964 + \epsilon)$-approximate for WTAP. For TAP this can be strengthened to $(5/3 + \epsilon)$.

Improving Adjiashvili's Algorithm

The Idea

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- Improve the guarantee in $(\star)$ by strengthening (P_1) through CG cuts!

Chvátal-Gomory Cuts for TAP

Recap: Bundle LP formulation for WTAP:

$$\begin{aligned} \min \quad & \sum_{l} w_l x_l && (\mathsf{P}_1) \\ \text{s.t.} \quad & \sum_{l \in \mathsf{cov}(e)} x_l \geq 1 \quad (e \in E) \\ & \sum_{l \in \mathsf{cov}(B)} w_l x_l \geq \mathsf{opt}_B \quad (B \in B_\gamma) \\ & x \geq \mathbb{0} \end{aligned}$$

Chvátal-Gomory Cuts for TAP

- The inequality

$$\sum_{e\in E} \lambda_e x(\mathsf{cov}(e)) + \sum_{l\in L} \mu_l x_l \geq \left\lceil \sum_{e\in E} \lambda_e \right\rceil, \qquad (\star)$$

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- Such cuts are valid for the IP corresponding to P_1.

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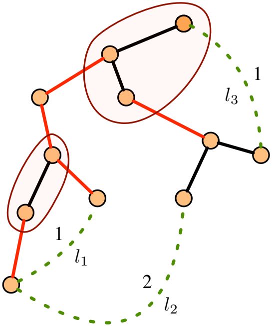
- Constraints ($\star$) have a nicer graphical representation.



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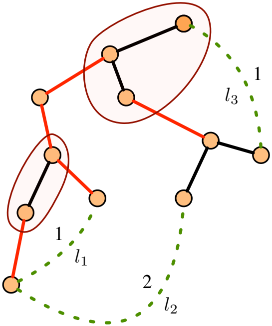

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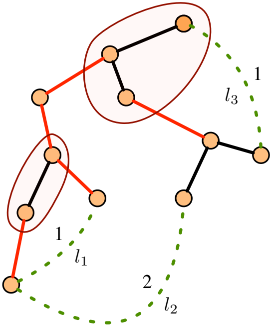

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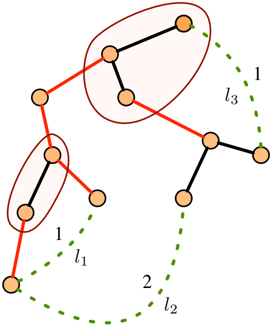

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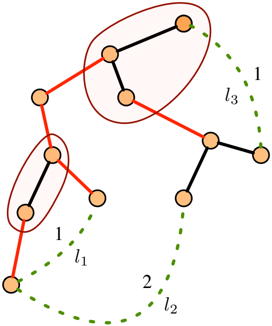
- For S with odd $|\delta(S)|$ we rewrite $(\star)$ as

$$x(\pi(S)) \geq (|\delta(S)| + 1)/2$$



Odd Cut Bundle LP

$$\min \quad \sum_{l} w_l x_l \tag{P$_2$}$$

$$\text{s.t.} \quad x(\pi(S)) \geq \frac{|\delta(S)|+1}{2} \quad (S \subseteq V, |\delta(S)| \text{ odd}) \tag{$\star$}$$

$$\sum_{l \in \mathsf{cov}(B)} w_l x_l \geq \mathsf{opt}_B \quad (B \in B_\gamma)$$

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Theorem [Caprara, Fischetti '96]

Inequalities ($\star$) can be separated efficiently, and hence the (P$_2$) can be solved efficiently as well for constant γ.

Rounding Method 2- Another Look

- Obtain the odd-cut LP by dropping the bundle constraints from (P_2)

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Consequences:

- We can compute a solution S of cost no more than

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- Replacing P_1 by P_2 in Adjiashvili's algorithm, x^i can be shown to be feasible for odd cut LP of the instance induced by T^i

Proving the KeyTheorem

- Define matrix M to be the incidence matrix of a bidirected graph if

$$\sum_i |M_{ij}| \leq 2,$$

for all j

$$\begin{bmatrix} 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}$$

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for all j

- Then B is a binet matrix if

$$B = R^{-1}S,$$

where $M = [SR]$ is the incidence matrix of a bidirected graph
  - with full row rank, and
  - R is a basis of M

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Proving the KeyTheorem

Theorem [Appa & Kotnyek '04,Edmonds & Johnson '70, '73]

For binet matrix $B \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^m$, the integer hull of

$$P = \{x : Bx \geq b, x \geq \mathbb{0}\}$$

is described by the $\{0, 1/2\}$ CG cuts.

Proving the KeyTheorem

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$\longrightarrow$ odd-cut LP (P_1) is integral in these cases

Improved Algorithm

Rounding Method 1

x feasible solution to (P_2), can compute S with

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Rounding Method 2

x feasible solution to (P_2), can compute S with

$$c(S) \leq 2\lambda \sum_{l \in \mathcal{I}} w_l x_l^i + \frac{4\lambda}{3(\lambda - 1)} \sum_{l \in \mathcal{C}} w_l x_l^i$$

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Thanks!